Computers and Mathematics with Applications 64 (2012) 115–120

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A quartic B-spline for second-order singular boundary value problems

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ARTICLE INFO

Article history:
Received 3 March 2011
Received in revised form 6 January 2012
Accepted 9 January 2012

Keywords:
Second-order singular boundary value problems
Quartic B-splines
Optimization

ABSTRACT

Second-order singular boundary value problems are treated using quartic (fourth-degree) B-spline approximations. The values of the coefficients, C_i, are chosen via optimization. Error analysis is discussed. The efficiency of the method is illustrated with several problems from the literature.

1. Introduction

Consider the following class of singular two-point boundary value problems:

$$x^{-k}(x^k y')' = f(x, y), \quad 0 < x \le 1 \tag{1a}$$

$$y'(0) = 0 \quad \text{and} \quad y(1) = \beta \tag{1b}$$

where β is a finite constant and $k \ge 1$. In order to ensure the existence and uniqueness of the solution of (1), we assume that $f(x, y)$ is continuous, $\frac{\partial f}{\partial y}$ exists and is continuous and, further, $\frac{\partial f}{\partial y} \ge 0$ [1–3].

Second-order singular boundary value problems occurs in several areas of applied mathematics and physics. Thus, they have been of interest to many authors [4–7]. Various methods have been applied for solving these kinds of problems, such as those in [8–10]. The use of cubic splines in solving second-order problems was first explored by Bickley [11] in 1968. Following this, Albasiny and Hoskins [12] obtained spline solutions by solving a tri-diagonal matrix system. Fyfe [13] examined the method suggested by Bickley [11] and carried out an error analysis. Goh et al. [14] used the cubic B-spline for solving one-dimensional heat and wave equations. Kumar [15] claimed that, in comparison with the finite difference method, the spline solution gives a simpler and more practical method for solving singular boundary problems. Since the problems are second-order ones, most of the researchers prefer third-degree (cubic) splines for solving them [16,17].

In this paper, a B-spline of higher degree is used to construct numerical solutions to second-order singular boundary problems. The values of the coefficients, C_i, $i = -2, -1, \ldots, n+1$, are obtained through optimization. Error analysis of the quartic B-splines is also presented. This method is then tested on a class of second-order singular boundary value problems to demonstrate its efficiency.

2. The quartic B-splines

In this paper, quartic B-spline functions are used to solve singular problems. Suppose a partition π of $[a, b]$ is equally divided by the knots x_i into n segments $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, where $a = x_0 < x_1 < \cdots < x_n = b$, such that

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doi:10.1016/j.camwa.2012.01.022

$x_i = a + ih$ and $h = (b - a)/n$. Then, the quartic B-spline functions can be defined via the following relationships [18]:

$$B_{4,i}(x) = \frac{1}{24h^4}\begin{cases}(x - x_{i-2})^4, & x \in [x_{i-2}, x_{i-1}]\\ h^4 + 4h^3(x - x_{i-1}) + 6h^2(x - x_{i-1})^2 + 4h(x - x_{i-1})^3 - 4(x - x_{i-1})^4 & x \in [x_{i-1}, x_i]\\ 11h^4 + 12h^3(x - x_i) - 6h^2(x - x_i)^2 - 12h(x - x_i)^3 + 6(x - x_i)^4, & x \in [x_i, x_{i+1}]\\ h^4 + 4h^3(x_{i+2} - x) + 6h^2(x_{i+2} - x)^2 + 4h(x_{i+2} - x)^3 - 4(x_{i+2} - x)^4, & x \in [x_{i+1}, x_{i+2}]\\ (x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}].\end{cases} \tag{2}$$

3. The numerical method for singular boundary value problems

For the linear case, the above equation, Eq. (1a), can be taken as [19]

$$y''(x) + \frac{k}{x}y'(x) + b(x)y(x) = c(x), \quad 0 < x \le 1 \tag{3a}$$

with the boundary conditions

$$y'(0) = 0 \quad \text{and} \quad y(1) = \beta. \tag{3b}$$

It can be noted that the above second-order differential equation (3a) is undefined when $x = 0$. Thus, the L'Hôpital rule is applied at the singular point, and then the boundary value problem can be transformed into the following form [16,9]:

$$\begin{cases}(k + 1)y''(x) + b(0)y(x) = c(0), & \text{for } x = 0,\\ y''(x) + \dfrac{k}{x}y'(x) + b(x)y(x) = c(x), & \text{for } x \neq 0.\end{cases} \tag{4}$$

The solution of Eq. (3a) is approximated by [20]

$$S(x) = \sum_{i=-2}^{n+1} C_i B_{4,i}(x) \tag{5}$$

where the C_i are the unknown real coefficients and the $B_{4,i}(x)$ are the basis functions of the quartic B-spline. Eq. (5) can be simplified to

$$S(x_i) = C_{i-2}B_{i-2}(x_i) + C_{i-1}B_{i-1}(x_i) + C_iB_i(x_i) + C_{i+1}B_{i+1}(x_i) \approx y(x_i), \tag{6a}$$

and we can obtain

$$S'(x_i) = C_{i-2}B'_{i-2}(x_i) + C_{i-1}B'_{i-1}(x_i) + C_iB'_i(x_i) + C_{i+1}B'_{i+1}(x_i) \approx y'(x_i), \tag{6b}$$

$$S''(x_i) = C_{i-2}B''_{i-2}(x_i) + C_{i-1}B''_{i-1}(x_i) + C_iB''_i(x_i) + C_{i+1}B''_{i+1}(x_i) \approx y''(x_i), \quad i = 0, 1, 2, \ldots, n. \tag{6c}$$

In order to get the approximations of Eqs. (3a)–(3b) at the point $x = x_i$, we substitute Eqs. (6a)–(6c) into Eqs. (4) and (3b); this leads to

$$\begin{cases}(k + 1)S''(0) + b(0)S(0) = c(0), & i = 0\\ S''(x_i) + \dfrac{k}{x_i}S'(x_i) + b(x_i)S(x_i) = c(x_i), & i = 1, 2, \ldots, n\end{cases} \tag{7}$$

$$S'(x_i) = 0 \quad \text{for } i = 0, \tag{8a}$$

$$S(x_i) = \beta \quad \text{for } i = n. \tag{8b}$$

A system of $(n + 3)$ equations with $(n + 4)$ unknowns $C_{-2}, C_{-1}, \ldots, C_{n+1}$ is thus obtained. This can be written in matrix–vector form as follows:

$$P\mathbf{C} = Q \tag{9}$$

where P is an $(n + 3) \times (n + 4)$-dimensional band matrix, $\mathbf{C} = [C_{-2}, C_{-1}, C_0, \ldots, C_{n+1}]^T$ and

$$Q = [y'(x_0), c(x_0), c(x_1), c(x_2), \ldots, c(x_n), y(x_n)]^T$$

where T denotes the transpose. This means that there are infinitely many solutions for C_i, $i = -2, -1, \ldots, n + 1$.

4. Optimization

By applying the Gauss–Jordan Elimination Method to our linear system (9), we can write the C_i, $i = -2, -1, \ldots, n$, in terms of C_{n+1}. Here, we consider the variable $\lambda = C_{n+1}$, which means that we have $C_i(\lambda), i = 0, 1, \ldots, n$, and thus for each interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, we have

$$S(x, \lambda) = C_{i-2}(\lambda)B_{i-2}(x) + C_{i-1}(\lambda)B_{i-1}(x) + C_i(\lambda)B_i(x) + C_{i+1}(\lambda)B_{i+1}(x) + C_{i+2}(\lambda)B_{i+2}(x). \tag{10}$$

Suppose that our solution $S(x, \lambda)$ is good enough to approximate the exact solution $y(x)$. Thus we expect that [21]

$$S''(x, \lambda) + \frac{k}{x}S'(x, \lambda) + b(x)S(x, \lambda) \approx c(x).$$

Therefore, the error of the approximation can be written as

$$E(x, \lambda) = S''(x, \lambda) + \frac{k}{x}S'(x, \lambda) + b(x)S(x, \lambda) - c(x). \tag{11}$$

Since the biggest error occurs at the midpoint in each interval $[x_i, x_{i+1}]$, the error $E(x, \lambda)$ is evaluated at $x_i* = (\frac{x_i+x_{i+1}}{2})$, $i = 0, 1, \ldots, n-1$. Hence, the expression $E(x_i*, \lambda)$ contains only one variable, λ. Now, we wish to minimize the error norm, the L_2-norm, such that

$$L_2 = \sqrt{\sum_{i=0}^{n-1} E(x_i*, \lambda)^2} = 0. \tag{12}$$

After solving Eq. (12), we can get the values of λ and C_i for $i = -2, -1, \ldots, n$. Thus, the solutions for each knot, x_i, can be approximated from Eq. (6a).

5. Error analysis

From the basis function (2) and the approximations (6a)–(6c), we can obtain the following:

$$S(x_i) = \left(\frac{1}{24}\right)C_{i-2} + \left(\frac{11}{24}\right)C_{i-1} + \left(\frac{11}{24}\right)C_i + \left(\frac{1}{24}\right)C_{i+1}$$

$$S'(x_i) = \left(-\frac{1}{6h}\right)C_{i-2} + \left(-\frac{1}{2h}\right)C_{i-1} + \left(\frac{1}{2h}\right)C_i + \left(\frac{1}{6h}\right)C_{i+1}$$

$$S''(x_i) = \left(\frac{1}{2h^2}\right)C_{i-2} + \left(-\frac{1}{2h^2}\right)C_{i-1} + \left(-\frac{1}{2h^2}\right)C_i + \left(\frac{1}{2h^2}\right)C_{i+1}$$

$$S'''(x_i) = \left(-\frac{1}{h^3}\right)C_{i-2} + \left(\frac{3}{h^3}\right)C_{i-1} + \left(-\frac{3}{h^3}\right)C_i + \left(\frac{1}{h^3}\right)C_{i+1}.$$

Then, the following relationships can be obtained:

$$h[S'(x_{i-2}) + 11S'(x_{i-1}) + 11S'(x_i) + S'(x_{i+1})] = 4[S(x_{i+1}) + 3S(x_i) - 3S(x_{i-1}) - S(x_{i-2})] \tag{13}$$

$$h^2S''(x_i) = 2[S(x_{i+1}) - 2S(x_i) + S(x_{i-1})] - \frac{h}{2}[S'(x_{i+1}) - S'(x_{i-1})] \tag{14}$$

$$h^3S'''(x_i) = 12[S(x_{i+1}) - S(x_{i-1})] - 3h[S'(x_{i+1}) + 6S'(x_i) + S'(x_{i-1})] \tag{15}$$

$$h^4S^{(4)}(x_{i+}) = 24[S(x_{i-1}) + S(x_i) - 2S(x_{i+1})] + 6h[S'(x_{i-1}) + 8S'(x_i) + 3S'(x_{i+1})] \tag{16a}$$

$$h^4S^{(4)}(x_{i-}) = 24[S(x_{i+1}) + S(x_i) - 2S(x_{i-1})] - 6h[S'(x_{i+1}) + 8S'(x_i) + 3S'(x_{i-1})] \tag{16b}$$

where $S^{(4)}(x_{i+})$ denotes the value of $S^{(4)}(x_i)$ in $[x_i, x_{i+1}]$. By using the operator notation $E(S(x_i)) = S(x_i)$, Eq. (13) can be written as [22]

$$h(E^{-2} + 11E^{-1} + 11 + E)S'(x_i) = 4(E + 3 - 3E^{-1} - E^{-2})S(x_i). \tag{17}$$

Since $E = e^{hD}$ where $D \equiv d/dx$, operator E can be written in an expansion form in powers of hD:

$$e^{hD} = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} + \frac{h^5D^5}{5!} + \cdots$$

$$e^{-hD} = 1 - hD + \frac{h^2D^2}{2!} - \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} - \frac{h^5D^5}{5!} + \cdots.$$

Therefore, the above form, Eq. (17), can be expressed as [22]

$$h(e^{-2hD}+11e^{-hD}+11+e^{hD})S'(x_i)=4(e^{hD}+3-3e^{-hD}-e^{-2hD})S(x_i) \tag{18}$$

or

$$24h\left(1-\frac{1}{2}hD+\frac{1}{3}h^2D^2-\frac{1}{8}h^3D^3+\frac{7}{144}h^4D^4-\cdots\right)S'(x_i)$$
$$=24h\left(D-\frac{1}{2}hD^2+\frac{1}{3}h^2D^3-\frac{1}{8}h^3D^4+\cdots\right)y(x_i). \tag{19}$$

Then, it can be simplified to

$$\begin{aligned}
S'(x_i) &= \left(D-\frac{1}{2}hD^2+\frac{1}{3}h^2D^3-\cdots\right)\left[1+\left(-\frac{1}{2}hD+\frac{1}{3}h^2D^2-\frac{1}{8}h^3D^3+\cdots\right)\right]^{-1}y(x_i)\\
&= \left(D-\frac{1}{2}hD^2+\frac{1}{3}h^2D^3-\cdots\right)\left[1-\left(-\frac{1}{2}hD+\frac{1}{3}h^2D^2-\cdots\right)+\left(-\frac{1}{2}hD+\frac{1}{3}h^2D^2-\cdots\right)^2-\cdots\right]y(x_i)\\
&= \left(D-\frac{1}{2}hD^2+\frac{1}{3}h^2D^3-\cdots\right)\left[1+\frac{1}{2}hD-\frac{1}{12}h^2D^2-\frac{1}{12}h^3D^3+\frac{11}{1440}h^5D^5+\cdots\right]y(x_i)\\
&= \left(D+\frac{1}{720}h^4D^5-\frac{1}{2016}h^6D^7+\frac{1}{17280}h^8D^9+\cdots\right)y(x_i).
\end{aligned}$$

Hence,

$$S'(x_i)=y'(x_i)+\frac{1}{720}h^4y^{(5)}(x_i)-\frac{1}{2016}h^6y^{(7)}(x_i)+\frac{1}{17280}h^8y^{(9)}(x_i)+O(h^{10}). \tag{20}$$

By using the same approach for Eqs. (14)–(16b), we can derive the following relations:

$$S''(x_i)=y''(x_i)-\frac{1}{240}h^4y^{(6)}(x_i)+\frac{1}{6048}h^6y^{(8)}(x_i)+O(h^8) \tag{21}$$

$$S'''(x_i)=y'''(x_i)-\frac{1}{12}h^2y^{(5)}(x_i)+\frac{1}{240}h^4y^{(7)}-\frac{1}{3024}h^6y^{(9)}+O(h^8) \tag{22}$$

$$S^{(4)}(x_i)=y^{(4)}(x_i)+\frac{1}{12}h^2y^{(6)}(x_i)-\frac{1}{720}h^4y^{(8)}(x_i)-\frac{1}{7560}h^6y^{(10)}+O(h^8) \tag{23}$$

$$S^{(5)}(x_i)=y^{(5)}(x_i)-\frac{211}{151200}h^6y^{(11)}(x_i)-\frac{421}{1814400}h^8y^{(13)}(x_i)+O(h^{10}). \tag{24}$$

Now, we define $e(x)=S(x)-y(x)$ and substitute the relations (20)–(24) into the Taylor series expansion of $e(x_i+\theta h)$ to obtain [23,19]

$$e(x_i+\theta h)=-\frac{(10\theta^2-1)\theta}{720}h^5y^{(5)}(x_i)+\frac{(5\theta^2-3)\theta^2}{1440}h^6y^{(6)}(x_i)+\frac{(7\theta^2-5)\theta}{10080}h^7y^{(7)}(x_i)+O(h^8). \tag{25}$$

Thus, the quartic uniform B-spline is $O(h^5)$ accurate.

6. Numerical results

In this section, a class of singular boundary value problems, which are discussed widely in the literature [16,9,17], are solved using the quartic B-spline.

6.1. Problem 1

Consider Bessel's equation of order 0:

$$y''(x)+\frac{1}{x}y'(x)+y(x)=0,$$
$$y'(0)=0, \qquad y(1)=1.$$

The exact solution for the problem is given by $y(x)=\frac{J_0(x)}{J_0(1)}$. The maximum absolute errors (L_∞-norms) and Euclidean norms (L_2-norms) for different values of the step size h are tabulated in Table 1 and compared with the results obtained by Caglar and Caglar [16].

Table 1
Comparison of error norms for problem 1 when using different values of h.

h	Cubic B-spline [16]		Proposed method	
	L_∞-norm	L_2-norm	L_∞-norm	L_2-norm
0.1	1.14094×10^{-4}	2.67891×10^{-4}	1.66706×10^{-6}	1.88608×10^{-6}
0.05	2.81966×10^{-5}	9.19744×10^{-5}	2.03647×10^{-7}	2.34234×10^{-7}
0.02	4.49361×10^{-6}	2.28986×10^{-5}	1.42101×10^{-8}	1.62257×10^{-8}
0.01	1.12263×10^{-6}	8.05488×10^{-6}	1.43704×10^{-9}	1.63884×10^{-9}

Table 2
Comparison of the computational results with the exact solutions for when $h = 0.05$.

x	Exact	Cubic B-spline [16]	Cubic spline [17]	HFDM [9]	Proposed method
0.0	3.257206	3.256908	3.256912	3.257208	3.257206
0.1	3.275624	3.275329	3.275332	3.275625	3.275624
0.2	3.331322	3.331030	3.331033	3.331323	3.331321
0.3	3.425641	3.425356	3.425360	3.425642	3.425641
0.4	3.560864	3.560589	3.560592	3.560864	3.560863
0.5	3.740271	3.740013	3.740016	3.740272	3.740271
0.6	3.968246	3.968011	3.968013	3.968247	3.968246
0.7	4.250393	4.250191	4.250193	4.250394	4.250393
0.8	4.593706	4.593551	4.593551	4.593706	4.593706
0.9	5.006766	5.006677	5.006677	5.006766	5.006766
1.0	5.500000	5.500000	5.500000	5.500000	5.500000

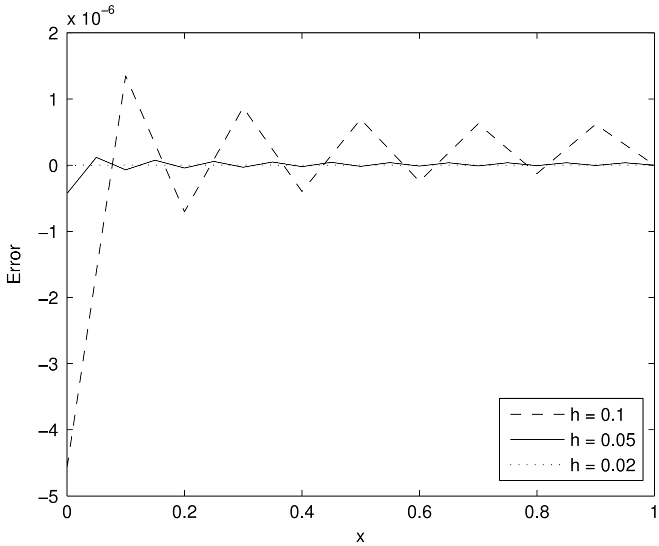

Fig. 1. Numerical errors for the quartic B-spline when using different values of h.

6.2. Problem 2

The following second-order differential equation and the boundary conditions are given:

$$y''(x) + \frac{2}{x}y'(x) - 4y(x) = -2, \quad 0 < x \le 1,$$
$$y'(0) = 0, \qquad y(1) = 5.5.$$

The exact solution is $y(x) = 0.5 + \frac{5\sinh 2x}{x\sinh 2}$. The approximations obtained for each knot, x_i, when $h = 0.05$ are presented in Table 2 and compared with those obtained in [16,9,17] and the exact solutions.

6.3. Problem 3

The boundary value problem is given by

$$y''(x) + \frac{1}{x}y'(x) = \left(\frac{8}{8-x^2}\right)^2,$$

$$y'(0) = 0, \qquad y(1) = 0.$$

The exact solution is known to be $y(x) = 2\log(\frac{7}{8-x^2})$. Fig. 1 depicts the errors obtained for different values of the step size h.

7. Conclusions

In this paper, the quartic uniform B-spline has been applied to construct the numerical solutions to a family of second-order singular boundary value problems. With one additional unknown, C_{n+1}, the approximations of the solutions can be optimized by minimizing the error norm. The numerical results show that the quartic B-spline can be used to approximate the exact solutions of the second-order singular two-point boundary value problems accurately.

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